Anomalous diffusions in option prices: connecting trade duration and the volatility term structure*

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Abstract

Anomalous diffusions arise as scaling limits of continuous-time random walks (CTRWs) whose innovation times are distributed according to a power law. The impact of a non-exponential waiting time does not vanish with time and leads to different distribution spread rates compared to standard models. In financial modelling this has been used to accommodate for random trade duration in the tick-by-tick price process. We show here that anomalous diffusions are able to reproduce the market behaviour of the implied volatility more consistently than usual Lévy or stochastic volatility models. We focus on two distinct classes of underlying asset models, one with independent price innovations and waiting times, and one allowing dependence between these two components. These two models capture the well-known paradigm according to which shorter trade duration is associated with higher return impact of individual trades.

We fully describe these processes in a semimartingale setting leading no-arbitrage pricing formulae, and study their statistical properties. We observe that skewness and kurtosis of the asset returns do not tend to zero as time goes by. We also characterize the large-maturity asymptotics of Call option prices, and find that the convergence rate is slower than in standard Lévy regimes, which in turn yields a declining implied volatility term structure and a slower decay of the skew.

Keywords: Anomalous diffusions, volatility skew term structure, derivative pricing, CTRWs, inverse Lévy subordinators, time changes, Lévy processes, subdiffusions, Beta distribution, triangular arrays.

1 Introduction

In quantitative finance, asset returns typically evolve according to Itô diffusions or Lévy-type models. From a microstructural point of view, these can be seen as scaling limits of continuous-

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time random walks (CTRWs) with exponentially distributed inter-arrival times. Instead, subordinating CTRWs to a renewal process whose waiting times obey a power law yields, in the scaling limit, an *anomalous diffusion*, namely a space-time propagation process where the particle spreads at a rate different from the classical diffusive case. The use of anomalous diffusions in financial models was pioneered by Mainardi et al. (2000) and Scalas et al. (2000), and they have proved useful to capture memory effects, trade idle time, and other microstructural price features exhibited by high-frequency time series.

However, applications of anomalous diffusions for continuous time option pricing have so far been scarce. The sub-diffusive Black-Scholes model was introduced in Magdziarz (2009) to capture asset staleness and periods of trade inactivity, but implications on pricing and implied volatilities were not illustrated. Cartea and Meyer-Brandis (2010) analysed the volatility surface of a CTRW whose innovation times are distributed according to a Mittag-Leffler hazard function; they produced explicit option pricing formulae, and provided evidence that the long-term skewness and smile can be captured.

We show here how anomalous diffusions in Equity returns can also capture the long-term behaviour of the implied volatility surface. Specifically, we argue that the persistence of a slowly decaying volatility skew can be explained by postulating the survival of trade durations effects at longer maturities. We consider returns and innovation times random walks which converge in the scaling limit to a pair of Lévy processes, one of which is a subordinator. According to Becker-Kern et al. (2004); Meerschaert and Scheffler (2008, 2010); Henry and Straka (2011); Jurlewicz et al. (2012), the associated CTRW subordinated to the renewal process of the innovation times converges to an anomalous diffusion which can be represented as a time-changed Lévy process. One appealing feature is that analytical formulae are known for the Laplace transforms (in the time variable) of the characteristic function of this limit, as well as integral expressions for the density functions in terms of the Lévy measures.

We analyse two distinct classes of anomalous diffusion models. The first is the purely *subdiffusive Lévy model* (SL), where the CTRW limiting diffusion consists of a Lévy process subordinated by an independent inverse-stable subordinator. In terms of the generating fractional Fokker-Planck equations such a class has been investigated in Cartea and del-Castillo-Negrete (2007). The particular case where the parent Lévy process is a Brownian motion was introduced in Magdziarz (2009); the compound Poisson case in Cartea and Meyer-Brandis (2010). Note also that the classic models in Mainardi et al. (2000), Scalas et al. (2000) admit a representation of this form. We revisit those as stochastic time changes, which is well suited for option pricing purposes. The time change representation of subdiffusive models also paves the way for our second second class of models, developing an idea from Becker-Kern et al. (2004). This novel asset price evolution realistically incorporates the dependence between the Lévy parent returns generating process and the inverse-stable subordinator modelling the trades waiting time. We call it the model with *dependent returns and trade duration* (DRD).

Apart from being natural outcomes of subordinated random walk tick-by-tick price models, these two models find are strongly supported by the econometric analysis by Engle (2000) and Dufour and Engle (2000), confirmed in numerous empirical studies later on. The evidence is that trading activity is inversely correlated with price impact, i.e. the 'volatility' of the asset price: the fewer the trades (longer duration), the more sluggish the price innovations; conversely, intense trading (short duration) is associated with higher price excursions. Remarkably, this principle is captured by the presented models.

We describe such Equity models in a semimartingale dynamic setting leading to no-arbitrage pricing relations under appropriate equivalent risk neutral measures. Using the results of Jurlewicz et al. (2012) on the Fourier-Laplace transforms of anomalous diffusions, we further provide familiar Parseval-Plancherel formulae for option prices in the spirit of Lewis (2001). Additionally, we study the moments and serial correlation properties of the model and show that skewness and kurtosis of the asset returns in the DRD model approach converge for large times, and do not vanish, contrary to Lévy models, leading in particular to profound differences on the long-term volatility smile.

Finally we characterize the large-maturity behaviour of Call option prices and find that the convergence rate is much slower than in standard Lévy or stochastic volatility regimes. We uncover a relationship according to which a declining implied volatility level implies a slowly– comparatively to Lévy and exponentially affine models–decaying skew. But we find that a (slowly) vanishing volatility level is a defining feature of these models, due to long-maturity prices converging much slower than in standard models. Ultimately, for the DRD model we show that the vanishing rate of the skew is slower than the usual $1/T$, in line with market data. As illustrated in the calibration in Section 8, the practical importance of anomalous diffusion model is that the 'duration parameter' β improves the cross-sectional fit to multiple maturities compared to a Lévy model, while having virtually no impact on the short-maturity calibration. This justifies the interpretation of β as a long term skew component.

We believe the contribution of this work to be manifold. We establish an explicit structural connection between trade duration and skew persistence; we introduce an analytical model that accounts for trades duration and dependence between trade waiting times and returns, consistent with Econometrics literature; we systematically unify the treatment of SL models under the umbrella of a single time-changed representation and the corresponding analytic pricing formulae; finally we extend the analysis of the 'Beta-time' process in Meerschaert and Scheffler (2004) and Jurlewicz et al. (2012), providing its moments and statistical properties through its time-changed representation.

In Section 2 we introduce fundamental building blocks and some useful notations. In Section 3 we introduce the CTRWs components of the base tick-by-tick model and the convergence theorem leading to the limiting continuous-time version. The anomalous diffusions are introduced in Section 4, together with their analytical properties and time-changed semimartingale representations, while their statistical properties are characterized in Section 5. In Section 6 we show how to construct equivalent pricing measures, and provide an integral price representation for European Call option prices. This allows us to study in Section 7 the structure of the corresponding implied volatility, with a particular emphasis on its large-maturity properties. Finally in Section 8, we numerically highlight interesting features of the SL and DRD models, and show that both models allow for a good fit to market data.

2 Foundational elements

We follow here (Kyprianou, 2014, Chapter 1). In a market filtration $(\Omega, \mathcal{F}_t, \mathbb{P})$ a Lévy process X is uniquely characterized by its Lévy exponent, namely the function $\psi_X : \mathbb{C} \to \mathbb{C}$ defined via the relation $\mathbb{E}\left[e^{-\mathrm{i}zX_t}\right] = \exp\left(-t\psi_X(z)\right)$, and given explicitly by the Lévy-Khintchine formula

$$\psi_X(z) = \mathrm{i}z\mu + \frac{z^2\sigma^2}{2} - \int_{\mathbb{R}} (e^{-\mathrm{i}zx} - 1 + \mathrm{i}zx\mathbb{1}_{|x|<1})\nu(dx), \tag{2.1}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$, and ν is a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx)$ is finite. A subordinator L is an almost surely non-decreasing Lévy process, with Lévy measure ν_L supported on $(0, \infty)$, and the Lévy-Khintchine representation for its Laplace exponent defined via the relation $\mathbb{E}\left[e^{-sX_t}\right] = \exp\left(-t\phi_L(s)\right)$, simplifies to

$$\phi_L(s) = s\mu - \int_0^\infty (e^{-su} - 1)\nu_L(du), \tag{2.2}$$

for $\mu > 0$, and where $\int_0^\infty u\nu_L(du) < \infty$. A bivariate Lévy process (X, L), with L as subordinator, has joint Fourier-Laplace transform $\mathbb{E}[e^{-\mathrm{i}zX_t - sL_t}] = \exp\left(-t\psi_{X,L}(z, s)\right)$ of the form

$$\psi_{X,L}(z, s) = \mathrm{i}z\mu_X + s\mu_L + \frac{z^2\sigma^2}{2} - \int_{\mathbb{R}}\int_0^\infty \left(e^{-\mathrm{i}zx - su} - 1 + \mathrm{i}zx\mathbb{1}_{|x|<1}\right)\nu_{X,L}(dx, du), \tag{2.3}$$

with Lévy-Laplace triplet $((\mu_X, \mu_T), \sigma, \nu_{X,T})$. If Y is a Lévy process, stochastic continuity implies that $Y_t = Y_{t-}$ almost surely for all $t > 0$, where Y_{t-} denotes the left limit. We write $\Delta Y_t := Y_t - Y_{t-}$. The first hitting time of $[t, \infty)$ of L is the random variable

$$H_t := \inf\left\{s > 0 \,|\, L_s > t\right\}. \tag{2.4}$$

which has continuous paths since L is strictly increasing, is $\mathcal{F}$-adapted by the Debut Theorem. The process H is called the *inverse-subordinator* of L. Of particular interest for us here is the case where L is an α-stable subordinator, i.e. $\psi_L(s) = s^\alpha$, $\alpha \in (0, 1)$, whose associated inverse-subordinator is central in fractional calculus and anomalous diffusions theory.

A *time change* is an increasing, almost surely finite process $(T_t)_{t\geq 0}$ diverging almost surely to infinity for large times. In particular, both L and H are time changes. If X is an $\mathcal{F}_t$-adapted semimartingale, then its time change by T is the $\mathcal{F}_{T_t}$-adapted semimartingale $(X_{T_t})_{t\geq 0}$. Further, if X is almost surely constant on all sets $[T_{t-}, T_t]$ we say that X is continuous with respect to T; in this case many other properties are preserved, and the semimartingale characteristics of X scale with T (Jacod, 1979, Chapter 10).

A *triangular array of random* variables is a collection of random variables $(Y_i^c, J_i^c)_{i\in\mathbb{N},c>0}$ indexed by a scale parameter c such that each $(Y_i^c)_{i\in\mathbb{N}}$ and $(J_i^c)_{i\in\mathbb{N}}$ is an iid sequence, but not necessarily independent from each other. For fixed c the variable Y_i^c retains the interpretation of the i-th price excursion, and J_i^c the time elapsed between two consecutive price moves. We can canonically associate to (Y_i^c, J_i^c) two families of continuous-time random walks (CTRWs):

$$R_t^c := \sum_{i=0}^{[t]} Y_i^c \qquad \text{and} \qquad T_t^c := \sum_{i=0}^{[t]} J_i^c, \tag{2.5}$$

and associate to T^c the counting process $N_t^c := \max\{n : T_n^c \leq t\}$. The notation $\widehat{\cdot}$ indicates the Fourier transform of probability measures, and the Laplace transform in the time variable is denoted by $\mathcal{L}(\cdot, s)$, where s is the new transformed variable.

3 The microstructural returns and their analytical properties

At a microscopic level, we postulate that the time series of returns and trade times, at the time scale c, are determined by a triangular array of random variables (Y_i^c, J_i^c), where Y_i^c determines the size of the returns implied by the equity price variation conditional to observing a price revision, and J_i^c dictates the time elapsed between subsequent revisions. The renewal process N^c corresponds then to the total number of price movements at t, and the tick-by-tick returns process Σ^c is thus given by subordinating R^c with N^c:

$$\Sigma_t^c := \sum_{i=0}^{N_t^c} Y_i^c. \tag{3.1}$$

At time t the price will have moved by a quantity $\sum_i^n Y_i^c$ if the n-th arrival time is recorded before t. Or, conditional to n price moves occurred by time s, the price will move again by Y_i^c before time $t > s$ if the waiting time variable J_{n+1}^c realizes at a value lesser than $t - s$. We assume that there exists a constant risk-free market rate $r > 0$ affecting the price growth linearly in time and independently of the time scale and modify (3.1) as

$$\Sigma_t^{c,*} := rt + \sum_{i=0}^{N_t^c} Y_i^c. \tag{3.2}$$

The reasons for this modification shall be explained further on. For the moment, we remark that this physical tick-by-tick model must be understood in the sense that only the price innovations correspond to market observations. Hence, the linear drift introduced in the random walk $\Sigma^{c,*}$ between two price movements does not give rise to a traded value, and impacts the price only at revisions time. However, further deterministic trends in the price dynamics, such as risk premia, are still possible and can be captured by an appropriate choice of Y^c.

3.1 Joint limits of CTRWs

The continuous-time pricing model we describe here is based on a scaling limit of the CTRW $\Sigma^{c,*}$ for an appropriately selected triangular array (Y_i^c, J_i^c). This setup encompasses classical mathematical finance models: when (Y_i^c) are centered i.i.ds with finite variance and $J_i^c = 1$ for all i, then the Central Limit Theorem yields a Brownian motion. If the Y_i^c have infinite variance and are in the domain of attraction of a stable process X, then their scaling limit yields exactly X. Considering random waiting times for J_c^i with finite expectation does not improve here the generality of the setting since by the Renewal Theorem $N_t^c \sim t/\mathbb{E}[J_i^c]$ in probability for large t.

Therefore, in order to build processes in which the trade time duration information has impact on the distribution of the scaling limit of Σ_t^c, one has to consider infinite-mean waiting times. Under this choice, taking the limit leads to an anomalous diffusion model for the asset price dynamics. The following result is central to the entire anomalous diffusions theory:

Theorem 3.1 (Becker-Kern, Meerschaert, Scheffler, Straka, Henry)**.** *Let (Y_i^c, J_i^c) be a triangular array of random variables and set R^c, T^c and Σ^c as in* (2.5)-(3.1). *If there exists a bivariate Lévy process (X, L) with L a subordinator, such that*

$$\lim_{c\uparrow\infty} (R_{ct}^c, T_{ct}^c) = (X_t, L_t)\,, \tag{3.3}$$

in the J_1-topology on the Skorokhod space $\mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$, then

$$\lim_{c\uparrow\infty} \Sigma_t^c \longrightarrow (X_{H_t-})^+, \tag{3.4}$$

in the J_1-topology on $\mathcal{D}(\mathbb{R})$, where $(X_{H_t-})^+$ is the right-continuous modification of X_{H_t-}.

This theorem has appeared in various forms and has an interesting evolution. It was first proved in Becker-Kern et al. (2004) under the weaker M1 topology, under an assumption only slightly weaker than independence between spatial evolution and waiting times. However, even if the process $(X_{H_t})_{t\geq 0}$ was claimed there to be the limit, the latter can be shown to coincide with $(X_{H_t-})_{t\geq 0}$ under such assumptions. This was remarked by Henry and Straka (2011), who also gave the full version of the theorem we use (i.e. allowing dependence), except that we allow the base processes to be CPPs. Another proof has been provided in (Jurlewicz et al., 2012, Theorem 3.1 and Theorem 3.5), this time including CPPs.

Remark 3.1. Unless the J_i^c are constant or exponentially distributed, the CTRW limit is not Markovian.

Example 3.1. For a sequence (Y_i) of iid centered random variables with unit variance, let $Y_i^c := c^{-1/2} Y_i$; consider further the i.i.d. sequence (J_i^c) distributed as $\mathrm{Exp}(\lambda)$, for some $\lambda > 0$. As previously detailed applying the Central Limit Theorem and the Renewal Theorem show the familiar convergence $\Sigma_t^c \to W_{\lambda t}$ for some Brownian motion W.

Example 3.2. Assume that (Y_i) and (J_i) are independent sequences of iid random variables belonging to the domain of attraction of respectively an α-stable law X, $\alpha \in (1,2)$, and a β-stable law L,$\beta \in (0,1)$, i.e. there exist regularly varying sequences (B_n) and (b_n) respectively with indices $-1/\alpha$, $-1/\beta$ such that $B_n \sum_{i=1}^n Y_i$ and $b_n \sum_{i=1}^n Y_i$ converge respectively to X and L almost surely. Then letting $Y_i^c := B(c)Y_i$ and $J_i^c := b(c)J_i$, with $B(c) := B_n \mathbb{1}_{\{t\in(C_{n-1},C_n]\}}$ and $b(c) := b_n \mathbb{1}_{\{t\in(c_{n-1},c_n]\}}$ yields an explicit triangular array and the theorem above applies with X_t and L_t being respectively the stable processes canonically associated with X and L. In this case, Theorem (3.1) collapses to (Meerschaert and Scheffler, 2004, Theorem 4.2).

Example 3.3. An explicit representation of the CGMY process as a CTRW limit can been obtained by appropriately tempering variables in the domain of attraction of a stable law, as explained in Chakrabarty and Meerschaert (2011). Combining this with Example 3.2 provides another explicit CTRW limit representation of (3.4) for a CGMY process X and a stable subordinator L.

3.2 Transform analysis and connections to fractional calculus

It is remarkable that the CTRW limit in Theorem 3.1 enjoys a very high degree of analytical tractability. For example the probability density of an inverse Lévy subordinator H is known in terms of the Lévy measure of L. Similarly, the law of X_{H_t-} can be recovered by integral transforms involving $\nu_{X,L}$ and the other Fourier-Laplace characteristics, as explained in Meerschaert and Scheffler (2008) and Jurlewicz et al. (2012). We recall the following from (Jurlewicz et al., 2012, Proposition 4.2):

Proposition 3.2. *Let X_{H_t-} be the CTRW limit in* (3.4)*, with law P_t. Then*

$$\mathcal{L}\left(\widehat{P}_t(dz), s\right) = \frac{1}{s}\frac{\phi_L(s)}{\psi_{X,L}(z,s)}. \tag{3.5}$$

The formula of the Laplace transform of X_{H_t-} is particularly simple. Having at hand a specification for X_{H_t-} in terms of the involved characteristic exponents, by virtue of (3.5) we are only one Laplace inversion away from the characteristic function, and we shall see that this inversion can be computed explicitly in our cases. From a theoretical perspective, the Fourier-Laplace transform of the process law provides an interesting connection between the stochastic representation of anomalous diffusions via CTRW limits, and the classical characterization of their laws as weak solutions of fractional abstract Cauchy problems. For details we refer the reader to Baumer et al. (2005); Meerschaert et al. (2013); Jurlewicz et al. (2012); Meerschaert and Scheffler (2008), and references therein.

4 The asset price models

We introduce here the two anomalous diffusions models we propose to establish the desired connection between trades duration and the implied volatility surface.

Definition 4.1. Let X be a Lévy process, L an independent β-stable subordinator, and (Y_i^c, J_i^c) a triangular array satisfying (3.3). We define the underlying price S as

$$S_t = S_0 \exp(rt + Y_t), \qquad S_0 > 0, \tag{4.1}$$

with $Y_t := X_{H_t-}$ given by (3.4), and shall consider the following two cases:

(SL) The *purely subdiffusive Lévy model* is such that (Y_i^c, J_i^c) satisfy the assumptions of Theorem 3.1 with (X, L) in the right-hand side of (3.3);

(DRD) The *model with dependent returns and trade durations* is such that (Y_i^c, J_i^c) satisfy the assumptions of Theorem 3.1 with (X_L, L) in the right-hand side of (3.3).

The two models are in appearance very similar, the only difference being that the second requires convergence of the return innovations to the subordinated Lévy process X_L instead of X. Yet, this difference is critical since this subordination is precisely what introduces coupling in the DRD model. We shall denote the return generating CTRW limit Y^{SL} and Y^{DRD} and, correspondingly, the price process as S^{SL} and S^{DRD} to distinguish the two processes. The underlying standard Lévy model is $S_t^0 = S_0 \exp(rt + X_t)$.

Remark 4.1. For the SL model, by independence and stochastic continuity of X, it is easy to see that $X_{H_t-} = X_{H_t}$ in law for each $t > 0$. From now on we shall use this operative definition.

Remark 4.2. As β tends to 1, L_t tends to t in probability and almost surely. Therefore the usual conditional independence argument (together with Proposition 4.2 below) shows that S_t tends in law to S_t^0. So in the limiting case, the Lévy models are recovered. Thus β can be interpreted as a parameter that regulates divergence from Lévy, and therefore quantifies the degree of 'anomaly' of the diffusion.

Example 4.1. When X is a Brownian motion and H an independent β-stable subordinator the resulting SL model is the *subdiffusive Black-Scholes* first introduced in Magdziarz (2009).

Example 4.2. Cartea and Meyer-Brandis (2010) introduce a CTRW model with independent trade duration and returns where the conditional waiting time is modelled through an hazard function. They in particular consider the latter to be of Mittag-Leffler type, i.e. $\mathbb{P}(T_n > t) = E_\beta(-t^\beta)$ (see also (6.2) further on), and the price innovations follow an arbitrary infinitely divisible distribution. The resulting driving CTRW is a Fractional Poisson process (FPP) Laskin (2003); Mainardi et al. (2004) with parameter β. Since an FPP can be represented as a CPP, time changed with an independent β-stable subordinator (as proved in Meerschaert et al. (2011)), the FPP model by Cartea and Meyer-Brandis (2010) is included in our framework.

Example 4.3. The original model in Scalas et al. (2000) and Mainardi et al. (2000) also admits an FPP representation, where the returns innovations follow a stable distribution, and can be written in terms of a triangular array limit (Meerschaert and Scalas (2004)).

Example 4.4. A comprehensive treatment of subdiffusive asset models obtained as fractional counterparts of popular Lévy models is provided in Cartea and del-Castillo-Negrete (2007), who tackle the option pricing problem by numerically solving the fractional partial differential equations characterizing their transition probabilities. In view of the results of (Meerschaert and Scheffler, 2008), all such models admit a time-changed representation of SL type.

Since the subordinator L has no drift, the sample paths of Y^{SL} and Y^{DRD} are Lebesgue almost everywhere constant (Bertoin, 1997, Chapter 2), and thus conveniently capture the idea of tick-by-tick trading and persistence of trade duration at all time scales. This also implicates that all equivalent measures for Y are mutually singular with respect to the usual diffusion processes. However, the discounted asset value necessarily contains a Lebesgue absolutely continuous part, orthogonal to all EMM for Y, coming from discounting by the market numeraire (the bank account). Therefore, in order for the Fundamental Theorem to apply, we need to cancel such part. This clarifies the choice (3.2) of modelling the interest rate effects externally to Y. Of course, nothing prevents that the physical dynamics Y itself have a drift in the component X. In Figure 1 we show sample paths of H and Y^{SL} when X is a standard Brownian motion, for two different values of β. As β increases, reversion to respectively the linear time and a standard Brownian return model with no trades duration effects is observed.

The non-Markovian structure of the two processes captures the possible memory effects in price formation when observing random waiting times between trades. As we shall see later, both the value of the process at time t and the time elapsed since the last price revision influence the price evolution. Dependence between trade times and price returns is a widely acknowledged fact, as pointed out in Engle and Russell (1998) and confirmed in several empirical studies. This makes the DRD model more realistic compared to the SL one, although the cost/benefit impact in terms of performance of embedding this feature remains to be assessed. For now, observe that the two models have the same number of parameters, so that modelling price/duration dependence does not add any dimension in the calibration and estimation.

It would be useful to find for the DRD a representation of $X^L_{H_t-}$ in terms of an independent time change similar to the one for the SL model. Consider first the special case $X = L$ in Theorem 3.1. Then L_{H_t-} is an $\mathcal{F}_{H_t}$-adapted time change. Indeed, the following proposition establishes that the DRD return model can be written as a time change with respect to L_{H_t-}.

Proposition 4.2. *Denote $X^L_t = X_{L_t}$ and $L^H_t = L_{H_t-}$. Then $(X_{L^H_t})_{t\geq 0}$ has a right-continuous modification which is a version of $(X^L_{H_t-})_{t\geq 0}$.*

Proof. Conditioning and using Fubini's Theorem yields, for $n \geq 1$, and any $t_1, \ldots, t_n$,

$$
\begin{aligned}
&\mathbb{P}(X^L_{H_{t_1}-} \in dx_1, \ldots, X^L_{H_{t_n}-} \in dx_n) \\
&= \int_{\mathbb{R}^n_+} \mathbb{P}(X^L_{s_1-} \in dx_1, \ldots, X^L_{s_n-} \in dx_n | H_{t_1} = s_1, \ldots, H_{t_n} = s_n) \mathbb{P}(H_{t_1} \in ds_1, \ldots, H_{t_n} \in ds_n) \\
&= \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} \mathbb{P}(X_{u_1} \in dx_1, \ldots, X_{u_n} \in dx_n | L_{s_1-} = u_1, \ldots, L_{s_n-} = u_n, H_{t_1} = s_1, \ldots, H_{t_n} = s_n) \right. \\
&\qquad \left. \mathbb{P}(L_{s_1-} \in du_1, \ldots, L_{s_n-} \in du_n | H_{t_1} = s_1, \ldots, H_{t_n} = s_n) \right) \mathbb{P}(H_{t_1} \in ds_1, \ldots, H_{t_n} \in ds_n) \\
&= \int_{\mathbb{R}^n_+} \mathbb{P}(X_{u_1} \in dx_1, \ldots, X_{u_n} \in dx_n | L_{H_{t_1}-} = u_1, \ldots, L_{H_{t_n}-} = u_n) \mathbb{P}(L_{H_{t_1}-} \in du_1, \ldots, L_{H_{t_n}-} \in du_n) \\
&= \mathbb{P}(X_{L^H_{t_1}} \in dx_1, \ldots, X_{L^H_{t_n}} \in dx_n). \qquad (4.2)
\end{aligned}
$$

Hence $X_{L^H_t}$ is a version of $X^L_{H_t-}$. To verify the existence of a right-continuous modification of $X_{L^H_t}$, observe that by stochastic continuity of X it suffices to show the existence a right-continuous modification of L^H. We observe that the values at which L^H is discontinuous are exactly the points in the image $\mathcal{R}$ of L which are isolated on the right. But since L has no drift, $\mathbb{P}(t \in \mathcal{R}) = 0$ by (Bertoin, 1997, Chapter 1, Proposition 1.9), so that replacing L^H with its right limits generates a càdlàg modification of L^H. □

There are then two ways of looking at the DRD returns process. The definition gives us a *dependent* representation using a *continuous* time change; Proposition 4.2 produces instead an *independent* representation employing a *discontinuous* time change. Both will be useful in the sequel. Proposition 4.2 will be used throughout without further mention.

Let us briefly describe the nature of the process L^H. It is easy to show that, for any $t \geq 0$,

$$L^H_t = \sup\{s < t : s = L_u, \text{ for some } u \geq 0\}. \qquad (4.3)$$

In light of this identification, the process L^H_t is sometimes called the *last sojourn process* and plays an important role in potential theory for Lévy processes. It is an increasing process (Bertoin (1997)) which tracks the largest value attained by L before leaving any given fixed level set $[0, t]$. When (the right-continuous version of) L^H_t jumps, its post-jump value is exactly t, and in any case $L^H_t \leq t$ almost surely. This ties in with the interpretation of L^H_t as a delayed calendar time. The following fact has been already remarked in Becker-Kern et al. (2004) and Jurlewicz et al. (2012), but we reformulate it in our context. We denote by $\mathcal{B}_{\alpha,\beta}$ the Beta distribution with parameters α and β.

Proposition 4.3. *For any $t \geq 0$, L^H_t is distributed as $t\mathcal{B}_{\beta,1-\beta}$.*

Proof. By definition of the Beta distribution, for $0 < y < t$,

$$\mathbb{P}(t\mathcal{B}_{\beta,1-\beta} < y) = \begin{cases} \displaystyle\int_0^y \frac{x^{\beta-1}(t-x)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)} dx, & \text{if } y \in (0, t), \\ 0, & \text{if } y \leq 0, \\ 1, & \text{if } y \geq t. \end{cases} \qquad (4.4)$$

We show then that the Fourier-Laplace transform of

$$p_t(x) := \frac{x^{\beta-1}(t-x)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)} \mathbb{I}_{\{x \le t\}} \tag{4.5}$$

satisfies Proposition 3.2 with $X = L$. First of all $\phi_L(s) = s^\beta$, and an easy computation produces $\psi_{L,L}(s,z) = (s - \mathrm{i}z)^\beta$, so that by (3.5) we need to verify

$$\mathcal{L}\left(\widehat{p}_t(z), s\right) = \frac{s^{\beta-1}}{(s-\mathrm{i}z)^\beta}. \tag{4.6}$$

The Fourier transform of p_t then reads

$$\widehat{p}_t(z) = \frac{1}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t e^{\mathrm{i}zx} x^{\beta-1}(t-x)^{-\beta} dx \tag{4.7}$$

where the integral is the convolution $f * g_z(t)$, where $f(u) = u^{-\beta}$ and $g_z(u) = u^{\beta-1}e^{\mathrm{i}zu}$. Since $\mathcal{L}(f * g_z, s) = \mathcal{L}(f, s)\mathcal{L}(g_z, s)$, with $\mathcal{L}(f, s) = s^{\beta-1}\Gamma(1-\beta)$, and $\mathcal{L}(g_z, s) = (s - \mathrm{i}z)^{-\beta}\Gamma(\beta)$, the statement follows. □

This proposition underpins the greater analytic tractability of the DRD model with respect to the SL model: somewhat paradoxically, the more realistic model is also the more tractable. Proposition 4.3 clarifies how the DRD model captures the paradigm of Engle (2000) and Dufour and Engle (2000). The DRD time-changed evolution obeys a form of delayed calendar time whose mass in $[0, t]$ concentrates more around 0 or t depending on whether β is close to zero or one (Figure 3). This mass represents the quantity of delay one has to apply to X to obtain the current price value. When L has a low β, that is when duration of trade is higher, the price evolution is stickier, since $\mathcal{B}_{\beta,1-\beta}t$ is much smaller than t with high probability. This is associated with a reduced impact of the individual trades on the price process because the informational content of sporadic trading is low. Conversely, as $\mathcal{B}_{\beta,1-\beta}t$ is close to t with high probability (namely when β is close to one) we observe a higher trading activity, typically associated with the presence of informed traders. In such a case the contribution of each single trade to the process of price formation is greater, and the impact of trading on price higher. A similar reasoning applies to the SL model. Here combining subordination with independence 'delays' the evolution of X for the time necessary to the next price revision to happen, but the resulting move retains the variance of an earlier point-in-time position of the process X. Therefore, again, the lower the β, the stickier the price dynamics.

5 Moments and time series properties

We derive some statistical properties of the SL and DRD models and provide some initial insight on the structure of the volatility surface they generate, anticipating the full analysis in Section 7. We begin with the moments of the DRD model, whose analytic tractability plays a major role.

The following proposition extends (Leonenko et al., 2014, Theorem 2.1) to higher cumulants. In this section, X is a given Lévy process, T an independent time change, and we let κ_i and τ_i denote their respective i-th cumulants, which we assume to exist for $i = 1, \dots, 4$.

Proposition 5.1. *The process $Y := X_T$ has moments up to order four, and its cumulants read*

$$\begin{aligned} \kappa_1^Y &= \tau_1\kappa_1, & \kappa_2^Y &= \tau_1\kappa_2 + \kappa_1^2\tau_2, \\ \kappa_3^Y &= \tau_1\kappa_3 + 3\kappa_1\kappa_2\tau_2 + \kappa_1^3\tau_3, & \kappa_4^Y &= (3\kappa_2^2 + 4\kappa_1\kappa_3)\tau_2 + 6\kappa_1^2\kappa_2\tau_3 + \kappa_4\tau_1 + \kappa_1^4\tau_4. \end{aligned} \tag{5.1}$$

Proof. In our notation $\kappa_n = -\left(\mathrm{i}^n\psi_X^{(n)}(0)\right)$. The usual conditioning argument yields

$$\mathbb{E}[Y_t] = \mathrm{i}\frac{d}{dz}\mathbb{E}\left[e^{-\mathrm{i}zY_t}\right]\Big|_{z=0} = \mathrm{i}\frac{d}{dz}\mathbb{E}\left[e^{-\psi_X(z)T_t}\right]\Big|_{z=0} = -\mathrm{i}\psi'_X(0)\mathbb{E}[T_t], \tag{5.2}$$

which gives κ_1^Y. Next

$$\mathbb{E}[Y_t^2] = -\frac{d^2}{d^2z}\mathbb{E}\left[e^{-\mathrm{i}zY_t}\right]\Big|_{z=0} = \psi''_X(0)\mathbb{E}[T_t] - \psi'_X(0)^2\mathbb{E}\left[T_t^2\right], \tag{5.3}$$

Subtracting from (5.3) the square of (5.2) reconstructs τ_2 and yields κ_2^Y. Similarly,

$$\mathbb{E}[Y_t^3] = -\mathrm{i}\frac{d^3}{d^3z}\mathbb{E}\left[e^{-\mathrm{i}zY_t}\right]\Big|_{z=0} = -\psi'''_X(0)\mathbb{E}[T_t] + 3\mathbb{E}\left[T_t^2\right]\psi'_X(0)\psi''_X(0) + \mathrm{i}\psi'_X(0)^3\mathbb{E}\left[T_t^3\right]; \tag{5.4}$$

calculating $\mathbb{E}[Y_t^3] - 3\mathbb{E}[Y_t]\mathbb{E}[Y_t^2] + 2\mathbb{E}[Y_t]^3$ and factoring the τ_i as necessary we obtain κ_3^Y. The last term κ_4^Y is obtained analogously. □

The above proposition confirms the well-known fact that a Lévy model X subordinated by a Lévy process L creates non-zero skewness and kurtosis even in presence of a mesokurtic and symmetric parent process X such as a Brownian motion. Our situation here is identical, and carries the message that trade duration alone can be a determinant of departure from normality of returns (thus, in an option pricing perspective, creating volatility smile). However, the moments term structure analysis is completely different. The key fact is that the moment time dispersion of a time-changed Lévy process only depends on the moments of the time change, and not those of X. In the usual Lévy subordination case, that is when T is a Lévy process, one then sees that the moment term structure is linear in t consistently with the fact that the subordinated process is itself Lévy. As a consequence returns skewness and kurtosis vanish with time. In contrast, specializing to our framework, produces a highly nonlinear moments time evolution. We analyse this in detail for the DRD model, where such evolution is polynomial in the cumulant degree.

Proposition 5.2. *For any $t \geq 0$, the first four cumulants of Y_t^{DRD} are*

$$
\begin{aligned}
\kappa_1^Y &= \beta\kappa_1 t,\\
\kappa_2^Y &= \beta\kappa_2 t + \frac{\kappa_1^2}{2}(1-\beta)\beta t^2,\\
\kappa_3^Y &= \beta\kappa_3 t + \frac{3\kappa_1\kappa_2}{2}(1-\beta)\beta t^2 - \frac{\kappa_1^3}{3}(1-\beta)\beta(2\beta-1)t^3,\\
\kappa_4^Y &= \beta\kappa_4 t + \frac{4\kappa_1\kappa_3 + 3\kappa_2^2}{2}\beta(1-\beta)t^2\\
&\quad -2(1-\beta)\beta(2\beta-1)\kappa_1^2\kappa_2 t^3 + \tfrac{\kappa_1^4}{8}(1-\beta)\beta\left(2-11\beta(1-\beta)\right)t^4,
\end{aligned} \tag{5.5}
$$

and the following asymptotic relations hold:

$$
\begin{aligned}
\lim_{t\uparrow\infty}\operatorname{Skew}(Y_t) &= \frac{2\sqrt{2}}{3}\frac{1-2\beta}{\sqrt{(1-\beta)\beta}}\operatorname{sgn}(\kappa_1), & \lim_{t\uparrow\infty}\operatorname{Kurt}(Y_t) &= \frac{1}{\beta(1-\beta)} - \frac{11}{2},\\
\lim_{t\downarrow 0}\sqrt{t}\,\operatorname{Skew}(Y_t) &= \frac{\kappa_3}{\sqrt{\beta\kappa_2^3}}, & \lim_{t\downarrow 0} t\,\operatorname{Kurt}(Y_t) &= \frac{\kappa_4}{\beta\kappa_2^2}.
\end{aligned} \tag{5.6}
$$

Proof. By explicitly integrating (4.5) we have the central moments of T_t:

$$\mu_1^T = \mathbb{E}\left[L_t^H\right] = \beta t = \tau_1 \tag{5.7}$$

$$\mu_2^T = \mathbb{V}\left[L_t^H\right] = \frac{1}{2}(1-\beta)\beta t^2 = \tau_2 \tag{5.8}$$

$$\mu_3^T = \mathbb{E}\left[(L_t^H - \tau_1)^3\right] = -\frac{1}{3}(1-\beta)\beta(2\beta-1)t^3 = \tau_3 \tag{5.9}$$

$$\mu_4^T = \mathbb{E}\left[(L_t^H - \tau_1)^4\right] = \frac{\beta}{8}(1-\beta)\left(2-11(1-\beta)\beta)\right)t^4 = \tau_4 + 3\tau_2^2. \tag{5.10}$$

Solving for τ_i and substituting in (5.1) yields (5.5). Calculating further the normalized cumulants $\operatorname{Skew}(Y_t) = \kappa_3^Y/(\kappa_2^Y)^{3/2}$ and $\operatorname{Kurt}(Y_t) = \kappa_4^Y/(\kappa_2^Y)^2$ and considering respectively the limit for large t and the leading order around $t = 0$ imply the limits in the proposition. □

In line with Remark 4.2, for $\beta = 1$ the non-normalized Lévy cumulants of X_t are recovered. In the DRD model, as the time scale gets larger, higher moments do not vanish, but converge to a level that only depends on β, and not on the value of the Lévy cumulants (the sign of κ_1 dictates the sign of the skewness). As frequently noted, returns leptokurtosis and negative skewness are important drivers of implied volatility smiles. It thus makes sense to deduce that non-zero time limits of skewness and excess kurtosis determine persistence of the volatility smile over time. In contrast, for t close to zero, moment explosions are observed, as in the Lévy case; the rate of this explosion is exactly that of exponential Lévy models, including–up to a normalization by β–the constant factor. This suggests that the short-term smile/skew behaviour of the DRD implied volatility should be identical to that of the underlying Lévy model. We will verify these intuitions and make the matters more precise in Section 7.

The analysis of the return series properties stems from the observation that the models we are studying, although not Markovian with respect to their own filtration, admit a Markovian embedding. Remarkably, for the DRD process the finite dimensional-distributions of such an embedded process is known. For any $t \geq 0$, we define the backward renewal time

$$V_t := t - L_t^H, \tag{5.11}$$

which represents the time elapsed from the current instant t to the previous price move. Knowing the price at t and the time since the last price move is enough to fully describe the law of the future asset evolution.

Proposition 5.3. *The following properties hold:*

(i) the pairs (Y^{SL}, V) and (Y^{DRD}, V) are time-homogeneous Markov processes;

(ii) the process Y^{SL} has correlated increments, whereas Y^{DRD} has uncorrelated increments;

(iii) the increments of Y^{SL} are non-stationary, whereas the increments of Y^{DRD} are (wide-sense) stationary;

Proof. Item (i) is proved in (Meerschaert and Straka, 2014, Theorem 4.1). For the SL model, statement (ii) can be deduced from (Leonenko et al., 2014, Example 3.2, Equation 9), since in our case $\mathbb{E}[X_1] \neq 0$. In the case of the DRD model, for $s \leq t$, we can write (we drop the model superscript for convenience)

$$\mathbb{E}[X_t X_s] = \mathbb{E}[(X_t - X_s)X_s] + \mathbb{E}\left[X_s^2\right] = (t-s)s\mathbb{E}[X_1]^2 + s\mathbb{V}[X_1] + s^2\mathbb{E}[X_1]^2 = ts\mathbb{E}[X_1]^2 + s\mathbb{V}[X_1],$$

so that by independence and conditioning

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_s) &= \mathbb{E}\left[L_t^H L_s^H\right]\mathbb{E}[X_1]^2 + \mathbb{E}\left[L_s^H\right]\mathbb{V}[X_1] - \mathbb{E}\left[L_t^H\right]\mathbb{E}\left[L_s^H\right]\mathbb{E}[X_1]^2 \\ &= \mathrm{Cov}\left(L_t^H, L_s^H\right)\mathbb{E}[X_1]^2 + \mathbb{E}\left[L_s^H\right]\mathbb{V}[X_1]. \end{aligned} \tag{5.12}$$

Thus, considering increments and using the above, together with Proposition 5.1,

$$\begin{aligned} \mathrm{Cov}(Y_t - Y_s, Y_s) &= \mathrm{Cov}(Y_t, Y_s) - \mathbb{V}[Y_s] = \mathbb{E}[X_1]^2\left(\mathrm{Cov}(L_t^H, L_s^H) - \mathbb{V}[L_s^H]\right) \\ &= \mathbb{E}[X_1]^2\mathrm{Cov}(L_t^H - L_s^H, L_s^H), \end{aligned} \tag{5.13}$$

so absence of returns autocorrelation is equivalently checked on L_t^H. Now (Meerschaert and Straka, 2014, Example 5.4) give the conditional transition probabilities $p_t(y_0, v_0, dy, dv) := \mathbb{P}(L_t^H \in dy, V_t \in dv \mid y_0, v_0)$ of the Markov process (Y_t, V_t) as:

$$p_t(y_0, 0, dy, dv) = \frac{v^{-\beta}}{\Gamma(1-\beta)}\frac{(t-v)^{\beta-1}}{\Gamma(\beta)}\delta_{y_0+t-v}(dy)dv\mathbb{1}_{\{0<v<t\}}, \tag{5.14}$$

$$\begin{aligned} p_t(y_0, v_0, dy, dv) &= \delta_{y_0}(dy)\delta_{v_0+t}(dv)\left(\frac{v_0+t}{v_0}\right)^{-\beta} \\ &\quad + \left(\int_{v_0}^{v_0+t}\left(\frac{v}{v_0}\right)^{-\beta}\delta_{v_0+y_0+t-v}(dy)\frac{(v_0+t-s-v)^{\beta-1}}{\Gamma(\beta)}\frac{\beta s^{-\beta-1}}{\Gamma(1-\beta)}ds\right)dv. \end{aligned} \tag{5.15}$$

Explicitly integrating the second line we have

$$p_t(y_0, v_0, dy, dv) = \delta_{y_0}(dy)\delta_{v_0+t}(dv)\left(\frac{v_0+t}{v_0}\right)^{-\beta} + \delta_{v_0+y_0+t-v}(dy)\left(\frac{t-v}{v}\right)^{\beta}\frac{(t-v+v_0)^{-1}}{\Gamma(\beta)\Gamma(1-\beta)}dv, \tag{5.16}$$

whence, for $t_2 > t_1$ the the joint probability densities P_{t_1,t_2} for $(L^H_{t_1}, V_{t_1}, L^H_{t_2}, V_{t_2})$ can be obtained through the Chapman-Kolmogorov equation

$$P_{t_1,t_2}(dy_1, dv_1, dy_2, dv_2) = p_{t_1}(0, 0, dy_1, dv_1)p_{t_2-t_1}(y_1, v_1, dy_2, dv_2). \tag{5.17}$$

Integrating out dv_1 and dv_2 from the explicit form of the above for $0 < v_1 < t_1 - y_1$, $0 < v_2 < t_2 - y_2$ leads to the joint density of $(L^H_{t_1}, L^H_{t_2})$:

$$\begin{aligned} P_{t_1,t_2}(dy_1, dy_2) =& \frac{y_1^{\beta-1}((t_1-y_1)(t_2-y_2))^{-\beta}(y_2-t_1)^{\beta}}{(\Gamma(1-\beta)\Gamma(\beta))^2(y_2-y_1)}\mathbb{1}_{\{0<y_1<t_1<y_2<t_2\}}dy_1dy_2 \\ &+ \frac{(t_2-y_1)^{-\beta}y_1^{\beta-1}}{\Gamma(1-\beta)\Gamma(\beta)}\delta_{y_2}(dy_1)dy_2. \end{aligned} \tag{5.18}$$

Setting $t_1 = t$ and $t_2 = t + h$, a long integration yields

$$\begin{aligned} &\mathrm{Cov}(L^H_{t+h}, L^H_t) = \int_{\mathbb{R}^+\times\mathbb{R}^+} y_1y_2P_{t,t+h}(dy_1, dy_2) - \beta^2 t(t+h) \\ &= \int_t^{t+h}\int_0^t \frac{y_1^{\beta-1}((t_1-y_1)(t_2-y_2))^{-\beta}(y_2-t_1)^{\beta}}{(\Gamma(1-\beta)\Gamma(\beta))^2(y_2-y_1)}dy_1dy_2 + \int_0^t \frac{(t+h-y_1)^{-\beta}y_1^{\beta+1}}{\Gamma(1-\beta)\Gamma(\beta)}dy_1 - \beta^2 t(t+h) \\ &= \frac{1}{2}t\beta(t+2h\beta+t\beta) - \beta^2 t(t+h) = \frac{1}{2}t^2(1-\beta)\beta = \mathbb{V}[L^H_t], \end{aligned} \tag{5.19}$$

and therefore $\mathrm{Cov}(L^H_{t+h} - L^H_t, L^H_t) = \mathrm{Cov}(L^H_{t+h}, L^H_t) - \mathbb{V}[L^H_t] = 0$, which shows that the increments of the DRD model are uncorrelated, and (ii) holds.

Finally, using (Meerschaert and Scheffler, 2004, Corollary 3.3) together with a conditional argument, we see that the expected value of the increments of Y^{SL} depends on t, so that these cannot be stationary. Combining $\mathbb{E}[Y^{DRD}_{t+h} - Y^{DRD}_t] = \mathbb{E}[X_1]\beta h$ with absence of correlation between increments in the DRD model finishes the proof of (iii). □

It is generally accepted that returns times series calculated at lags of above a couple of minutes show no autocorrelation. Stationarity is also a desirable statistical property shown by returns: both these stylized facts are captured by the DRD model, which in this respect is strikingly similar to a Lévy process. However, these properties are not featured by the SL model, further suggesting that the DRD might be preferable.

6 Measure changes and derivatives valuation

6.1 The physical measure and EMM transformations

In order to apply classical valuation theory, one needs to show that the physical dynamics admit a martingale specification and to identify (if possible) an explicit equivalent measure. In our models, there are two sources of market risk: the uncertainty in the returns distribution and the trade duration, captured respectively by the process X and L. We could in principle consider measure changes affecting the dynamics of both these processes. However, α-stable processes are not stable by equivalent measure change, since the Hellinger distance of the Lévy measures of any two stable subordinators is infinite. For example, if a standard Esscher transform is used, after measure change the process becomes tempered stable. Hence, since we are interested in the risk-neutral parametrizations of the SL and DRD models, we shall restrict our analysis to the class of EMM that only involve transformation of the law of X. As one may reasonably guess, such a class coincides with the set of equivalent measures under which the underlying Lévy model S^0 is itself a martingale.

Proposition 6.1. *Let S be of SL or DRD type, and $\mathbb{Q} \sim \mathbb{P}$ with $Z_t := d\mathbb{Q}/d\mathbb{P}$ a measure under which S^0 is a Lévy exponential martingale. Then both $\left(e^{-rt} S_t^{DRD}\right)_{t \geq 0}$ and $\left(e^{-rt} S_t^{SL}\right)_{t \geq 0}$ are $\mathcal{F}_{H_t}$-adapted martingales, respectively under $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, where $Z_{H_t} = d\widetilde{\mathbb{Q}}/d\mathbb{P}$.*

Proof. Assume S has SL dynamics and $S_0 = 1$. For any $t \geq 0$, let $S_t^{0,*} := \exp(rt + X_t^*)$, where $X_t^* = X_t + t\psi_X(\mathrm{i})$ are the risk-neutral dynamics of X_t under $\mathbb{Q}$. By independence the distribution of L_t under $\mathbb{Q}$ is the same as under $\mathbb{P}$. Since H is a continuous time change, X is continuous with respect to H, so by Kallsen and Shiryaev (2002) the cumulant process (Barndorff-Nielsen and Shiryaev, 2015, Chapter 4) of X_{H_t} coincides with $-H_t\psi_X(-z)$, and hence the process defined by

$$\exp(X_{H_t})\mathcal{E}(H_t\psi_X(\mathrm{i})) = \exp(X_{H_t} + H_t\psi_X(\mathrm{i})) = \exp(X_{H_t}^*) = e^{-rt} S_{H_t}^{0,*} \tag{6.1}$$

is a $\mathbb{Q}$-local martingale. By independence, taking expectation and conditioning on H_t, this process has expectation one, hence it is a true $\mathbb{Q}$-martingale. But as explained in (Fries and Torricelli, 2019, Lemma 5.1), time changing the $\mathbb{Q}$-dynamics of X_t^* by H_t is equivalent to applying the change of measure $\widetilde{\mathbb{Q}}$ to X_{H_t}, so that the claim follows. For the DRD model it suffices to observe that L_t^H is a bounded family of stopping times and thus $e^{-rt} S_t^{DRD} = \exp\left(X_{L_t^H}^*\right)$ is a $\mathcal{F}_{H_t}$-martingale under $\mathbb{Q}$ by Doob's Optimal Sampling Theorem. □

Again we emphasize that this is a subset of all the possible equivalent martingale measures and that for technical reasons we need to ignore a market price of duration risk. A model in which this risk can be priced can be obtained for example by considering for L_t the wider class of tempered stable subordinators, which is closed under the Esscher transform. This class, along with related questions of market completeness, is studied in Torricelli (2019); see also Fries and Torricelli (2019) for the situation when trade duration is caused by market suspensions.

6.2 The pricing formula

Having established that the risk-neutral specification comes in the form of a time-changed martingale exponential, Proposition 3.5 can be combined with standard integral price representations to yield semi-closed derivative valuation formulae. Remarkably, the characteristic functions of the log-price in the SL and DRD models admit a very simple representation in terms of the one-parameter Mittag-Leffler function

$$E_a(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+1)}, \tag{6.2}$$

where Γ is the usual Gamma function, and of the confluent Hypergeometric function

$${}_1F_1(a, b; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{6.3}$$

Theorem 6.2. *Let Y be either process in Definition 4.1, $F(\cdot)$ be a contingent claim on S maturing at T. Assume that $f(x) := F(e^x)$ is Fourier-integrable and let $\mathcal{S}_f$ be the domain of holomorphy of its Fourier transform $\widehat{f}$. Let $\Phi_t(z) := \mathbb{E}[e^{-\mathrm{i}zY_t}]$, denote with $\mathcal{S}_Y$ the domain of holomorphy of $\Phi_T(z)$, and assume $\mathcal{S}_f \cap \mathcal{S}_Y \neq \emptyset$. The price P_0 of the derivative paying $F(S_T)$ at time T is given by:*

$$P_0 = \mathbb{E}\left[e^{-rT} F(S_T)\right] = \frac{e^{-rT}}{2\pi} \int_{\mathrm{i}\gamma-\infty}^{\mathrm{i}\gamma+\infty} S_0^{-\mathrm{i}z} e^{-\mathrm{i}zrT} \Phi_T(z) \widehat{f}(z) dz. \tag{6.4}$$

The value γ is chosen such that the integration line lies in $\mathcal{S}_f \cap \mathcal{S}_Y$ and

$$\Phi_t(z) = \begin{cases} E_\beta\left(-\psi_X(z)t^\beta\right), & \text{if } Y_t = Y_t^{SL}, \\ {}_1F_1(\beta, 1, -t\psi_X(z)), & \text{if } Y_t = Y_t^{DRD}. \end{cases} \tag{6.5}$$

Proof. Under the given assumptions, the Plancherel representation (6.4) is standard (see Lewis (2001) for example), and we only need to prove (6.5). In the SL model, by independence of X and L we have $\psi(s, z) = \phi_L(s) + \psi_X(z) = s^\beta + \psi_X(z)$, and Proposition 3.2 then yields

$$\mathcal{L}(\Phi_t(z), s) = \frac{s^{\beta-1}}{s^\beta + \psi_X(z)}. \tag{6.6}$$

Inverting the right hand-side, as in Haubold et al. (2011), one obtains (6.5). In the DRD model after conditioning and applying Proposition 4.3, we obtain

$$\Phi_t(z) = \mathbb{E}\left[\exp\left(-\psi_X(z) L_t^H\right)\right] = \mathbb{E}\left[\exp\left(-t\psi_X(z)\mathcal{B}_{\beta,1-\beta}\right)\right], \tag{6.7}$$

and the statement follows from the characteristic function of $\mathcal{B}_{\beta,1-\beta}$. □

Remark 6.1. Fast computational routines for the Mittag-Leffler and the confluent hypergeometric functions are available in most software packages. Also, the two functions can be unified in a single software implementation by observing that the three-parameter Mittag-Leffler function

$$E_{a,b,c}(z) = \sum_{k=0}^{\infty} (c)_k \frac{z^k}{\Gamma(ak+b)} \tag{6.8}$$

is such that $E_{a,1,1}(z) = E_a(z)$ and $E_{1,1,c}(z) = {}_1F_1(c, 1, z)$. Furthermore when $a = b = c = 1$, then (6.8) reverts to the standard exponential, which is consistent with the fact that S^{SL} and S^{DRD} revert to the exponential Lévy model S^0.

Remark 6.2. The function E_β is entire and ${}_1F_1(\beta, 1, -t\psi_X(\cdot))$ is regular in the complex plane without the negative real axis; hence $\mathcal{S}_f \cap \mathcal{S}_Y \neq \emptyset$ depends on the domain of ψ_X and $\widehat{f}$ only.

Remark 6.3. If X_H has an FPP structure, then (6.4) coincides with the formula given by (Cartea and Meyer-Brandis, 2010, Theorem 3), when the jump sizes have infinitely divisible distribution.

One sees that the pricing formulae are formally obtained from the standard Lévy case by replacing the exponential function with two different kinds of 'stretched exponentials'. The parameter β relaxes the shape of the characteristic function, in particular in the tails, thereby generating large-maturity prices very different from the base case. This overcomes the 'curse of exponentiality' of the standard models (both Lévy and exponentially-affine), for which the long-maturity option prices follow Laplace-type asymptotics of leading order $\exp(-T)/\sqrt{T}$. We will detail this better, together with its implications on the volatility surface, in Section 7 below. Note that the two functions (6.2) and (6.3) have very different behaviours. In Figure 2, we can see for example that (6.2) has a cross-over region where its decay transitions from super to sub-exponential, whereas in (6.3), the integrand always dominates the exponential. This clearly as an immediate impact on the shape of the volatility surface, in Section 8.

7 Time asymptotics of the volatility surface

Bearing in mind the discussion so far, we naturally expect implications of trade duration (at least in the form we chose to model it) on the volatility surface. The anomalous diffusions we constructed are subdiffusions, i.e. have a slower spread rate than the benchmark Lévy models, hence a slower option price convergence for large maturity. That said, since Black-Scholes is a Lévy model, normalization with the Black-Scholes formula must generate a vanishing implied volatility term structure in order to match the slower price time evolution. Less intuitive is to find a reason why the long-term skew should decline slower than standard models. A first answer is provided by Section 5: skewness and kurtosis in our models do not vanish as time grows but converge to some strictly positive level. Therefore Gaussian returns aggregation is precluded, and time reversion to a flat volatility might be pushed further away in time[1]. However, as we shall show, an exhaustive answer is provided by the fact that both skew and level of the implied volatility are intimately connected, and the property of a null asymptotic implied volatility is sufficient to hamper the skew time decay.

Without loss of generality, we assume here $r = 0$ and $S_0 = 1$ and, with a slight abuse of notation denote $C(K,T)$ the Call option price with strike K and maturity T, and $C(k,T)$ its value as a function of $k = \log K$. In the Black-Scholes model $dS_t = \sigma S_t dW_t$, with $\sigma > 0$, the price of such a Call option is given by

$$C_{\mathrm{BS}}(K,T,\sigma) = S_0 \mathcal{N}\left(d\left(\sigma\sqrt{T}\right)\right) - K\mathcal{N}\left(d\left(\sigma\sqrt{T}\right) - \sigma\sqrt{T}\right),$$

[1] Gaussian aggregation is by no means responsible of the smile flattening, as shown by Rogers and Tehranchi (2010).

where $d(z) := -\frac{k}{z} + \frac{z}{2}$, $\mathcal{N}$ denotes the standard Gaussian cumulative distribution function, and n its derivative, the Gaussian density function. For $K, T \geq 0$, the implied volatility $\sigma(K,T)$ is the unique non-negative solution to $C(K,T) := C_{\mathrm{BS}}(K,T,\sigma(K,T))$, and the implied volatility skew is defined as

$$\mathcal{S}(K,T) := \frac{\partial \sigma}{\partial K}(K,T). \tag{7.1}$$

It is known by Rogers and Tehranchi (2010) that $\mathcal{S}(K,\cdot)$ converges to zero as the maturity increases, for each K. We begin with the following model-free lemma which, under some mild assumptions on the underlying distribution, connects the time decay of the skew with its level.

Lemma 7.1. *Let $(S_t)_{t\geq 0}$ be a martingale such that S_t is absolutely continuous in law for all t and converges to zero in distribution as t tends to infinity.*

(i) For any $K \geq 0$, if $\lim_{T\uparrow\infty} \sqrt{T}\sigma(K,T) = \infty$ then, as T tends to infinity,

$$\mathcal{S}(K,T) = \frac{2}{T\sigma(K,T)}\left(1 + \frac{2\log(K) - 4}{T\sigma(K,T)^2} + O(T^{-2}\sigma(K,T)^{-4})\right) - \frac{\mathbb{Q}(S_T \geq K)}{\sqrt{T}n(d(\sqrt{T}\sigma(K,T)))}; \tag{7.2}$$

(ii) as T tends to zero,

$$\mathcal{S}(1,T) = \sqrt{\frac{2\pi}{T}}\left(\frac{1}{2} - \mathbb{Q}(S_t \geq 1) - \frac{\sigma(1,T)\sqrt{T}}{2\sqrt{2\pi}} + O(\sigma^2(1,T)T)\right). \tag{7.3}$$

Proof. We only prove the first statement, as the second one is proved in (Gerhold et al., 2016, Lemma 2). Since S_t has an absolutely continuous law, then by (Figueroa-López et al., 2011, Lemma C.1), $\mathcal{S}$ exists and $\partial_K C(K,T) = -\mathbb{Q}(S_T \geq K)$. Therefore, applying the chain rule

$$\mathcal{S}(K,T) = -\frac{\partial_K C_{\mathrm{BS}}(K,T,\sigma(K,T)) + \mathbb{Q}(S_T \geq K)}{\partial_\sigma C_{\mathrm{BS}}(K,T,\sigma(K,T))}. \tag{7.4}$$

Set $z = \sqrt{T}\sigma(K,T)$. Using the formulae for the Black-Scholes Delta and Vega:

$$\mathcal{S}(K,T) = \frac{\mathcal{N}(-d(z)) - \mathbb{Q}(S_T \geq K)}{\sqrt{T}n(d(z))}. \tag{7.5}$$

Since

$$\mathcal{N}(x) = \frac{n(x)}{x}\left(1 - \frac{1}{x^2} + \mathcal{O}\left(x^{-4}\right)\right) \qquad \text{and} \qquad \frac{1}{d(x)} = \frac{2}{x} + \frac{4\log(K)}{x^3} + \mathcal{O}\left(x^{-5}\right),$$

as x tends to infinity, then

$$\frac{\mathcal{N}(-d(z))}{\sqrt{T}n(d(z))} = \frac{1}{\sqrt{T}d(z)}\left(1 - \frac{1}{d(z)^2} + \mathcal{O}\left(d(z)^{-4}\right)\right) = \frac{2}{\sqrt{T}z}\left(1 + \frac{2\log(K) - 4}{z^2} + \mathcal{O}\left((z^{-4}\right)\right),$$

and (7.2) follows by substituting z and combining the above with (7.5). □

Remark 7.1. If $S^0 = \exp(X)$ is a martingale for some Lévy process X, from the proof of Proposition 6.1, our models can be written as $S^0_{T_t}$ for some time change T_t, so that $S^0_{T_t}$ converges to zero almost surely as t tends to infinity, provided we know this to hold for S^0_t. Such a property for exponential Lévy models can be proved using fluctuations identities, since the assumption $\mathbb{E}[X_1] < 0$ implies (Bertoin, 1996, VI.4, Exercise 3) that X_t diverges to $-\infty$. A negative first moment is always the case for X_t when S^0 is a martingale, as it is apparent from the relations connecting the stochastic and the natural exponential (Barndorff-Nielsen and Shiryaev, 2015, Corollary 4.1). Regarding the absolute continuity of the price process, this follows from the fact that the law of the involved processes are weak solutions of fractional Cauchy problems. These can be found using arguments analogous to (Jurlewicz et al., 2012, Examples 5.2-5.4).

Part (i) of this lemma implicates that the level and skew of the implied volatility are entangled: one cannot modify the leading order $1/T$ of the skew decrease without postulating a zero or diverging asymptotic implied volatility level. In turn, a declining implied volatility can only be attained through a convergence rate of option prices distributions to the spot price slower than Gaussian, which is precisely the distinguishing feature of anomalous diffusions-based models.

Proposition 7.2. *As T tends to infinity, we have the following asymptotic expansions for the Call price $C(k,T)$, for any $k \in \mathbb{R}$:*

- *in the DRD model with $\beta \in (0,1]$, there exist C_β and $c_\beta > 0$ such that*

$$C(k,T) = 1 - \mathbb{1}_{\{\beta \neq 1\}} \frac{C_\beta}{\Gamma(1-\beta)} \frac{1}{T^\beta} \left(1 + \mathcal{O}\left(\frac{1}{T}\right)\right) - \frac{c^\beta}{\Gamma(\beta)} \frac{e^{-T\psi_X(\mathrm{i}/2)}}{T^{3/2-\beta}} \left(1 + \mathcal{O}\left(\frac{1}{T}\right)\right); \quad (7.6)$$

- *in the SL model with $\beta \in (0,1)$, there exists $C'_\beta > 0$ such that*

$$C(k,T) = 1 - \frac{C'_\beta}{\Gamma(1-\beta)} \frac{1}{T^\beta} \left(1 + \mathcal{O}\left(\frac{1}{T}\right)\right). \quad (7.7)$$

Proof. We begin from the price representation for a Call option

$$C(k,T) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{k\left(iu+\frac{1}{2}\right)}}{u^2 + 1/4} \Phi_T\left(u + \frac{i}{2}\right) du \quad (7.8)$$

which can be obtained from (6.4) by moving the integration contour inside the strip $\Im(z) = 1/2$ and applying the Residue Theorem (see Lewis 2001). In order to expand the function Φ_t, since the integration line contains points of variable argument, we must ensure that the Stokes phenomenon does not occur. Assume $\beta < 1$ in the DRD model. The asymptotic expansion of ${}_1F_1(a,b,z)$, for large $|z|$ is (Luke, 2012, Chapter 4):

$${}_1F_1(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(b-a)} z^{-a} e^{i\delta\pi a} {}_2F_0(a, 1+a-b, -z^{-1}) + \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z {}_2F_0(b-a, 1-a, z^{-1}) \quad (7.9)$$

with $\delta = 1$ if $\Im(z) > 0$ and $\delta = -1$ otherwise. So when $\Im(z) = 1/2$, since $\Re(\psi_X(z)) > 0$, for large T we have the well-defined asymptotic behaviour

$$
\begin{aligned}
{}_1F_1(\beta, 1, -T\psi_X(z)) \sim & \frac{(T\psi_X(z))^{-\beta}}{\Gamma(1-\beta)} {}_2F_0\left(\beta, \beta, (T\psi_X(z))^{-1}\right) \\
& + \frac{e^{-T\psi_X(z)}(-T\psi_X(z))^{\beta-1}}{\Gamma(\beta)} {}_2F_0\left(1-\beta, 1-\beta, -(T\psi_X(z))^{-1}\right). \quad (7.10)
\end{aligned}
$$

Now, since the integrand of the second summand of (7.8) is bounded by an integrable function, by dominated convergence we can take the limit as T tends to infinity of $C(k, T)$ under the integral sign. Now notice that as $|z|$ tends to infinity , $|\psi_X(z)|$ also tends to infinity because the risk-neutral drift of X must be nonzero by (Bertoin, 1997, Corollary 1.1.3). This implicates that along any line $\Im(z) = c$, $|\psi_X(z)|$ is strictly increasing. Also ψ_X is even in its real part and odd in its imaginary part, so that $|\psi_X(\cdot + \mathrm{i}/2)|$ must be an even function. We conclude that $|\psi_X(\cdot + \mathrm{i}/2)|$, has a positive minimum at the origin. Therefore we can replace the limit with (7.10) so long as T is larger than $1/|\psi_X(\mathrm{i}/2)|$. By truncating the series of ${}_2F_0$ at order 0 in the first summand of (7.10) and integrating, we attain the first term in (7.6) with

$$
C_\beta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{k(\mathrm{i}u+1/2)}}{(u^2 + 1/4)\psi_X(u + \mathrm{i}/2)^\beta} du. \quad (7.11)
$$

Regarding the exponential sub-leading terms we have to analyse the first order term

$$
I^\beta(T) := \int_{-\infty}^{\infty} \frac{e^{k(\mathrm{i}u+1/2)} e^{-T\psi_X(u+\mathrm{i}/2)}}{(u^2 + 1/4)\psi_X(u + \mathrm{i}/2)^{1-\beta}} du, \quad (7.12)
$$

which can be treated using the saddle point method as in Andersen and Lipton (2012). From the previous discussion, $\psi_X(\cdot)$ has a stationary point at $\mathrm{i}/2$, where the real part of the characteristic exponent has a minimum, so that for large T,

$$
I^\beta(T) \sim \frac{\sqrt{2\pi} 4 e^{k/2}}{\psi_X(\mathrm{i}/2)^{1-\beta} \sqrt{\psi_X''(\mathrm{i}/2) T}}, \quad (7.13)
$$

which yields the second term in (7.6) with

$$
c_\beta = \frac{4 e^{k/2}}{\psi_X(\mathrm{i}/2)^{1-\beta} \sqrt{2\pi \psi_X''(\mathrm{i}/2)}}. \quad (7.14)
$$

When $\beta = 1$ the whole proof collapses to the well-known steepest descent argument (Andersen and Lipton, 2012, Section 7) for the Lévy models price representation integral.

In the SL model we have, for any given $\beta < 1$, that so long as $\pi\beta/2 < \mu < \min\{\pi, \pi\beta\}$ the asymptotic series for E_β is given by (Haubold et al., 2011, Equation 6.5)

$$
E_\beta(z) = \begin{cases} \dfrac{e^{z^{1/\beta}}}{\beta} \displaystyle\sum_{k=1}^{n-1} \frac{1}{\Gamma(1-\beta k)} \frac{1}{z^k} + \mathcal{O}\left(z^{-n}\right), & \text{for } |\arg(z)| < \mu, \\ \displaystyle\sum_{k=1}^{n-1} \frac{1}{\Gamma(1-\beta k)} \frac{1}{z^k} + \mathcal{O}\left(z^{-n}\right), & \text{for } \mu < |\arg(z)| \leq \pi. \end{cases} \quad (7.15)
$$

Since $\Re(\Psi_X(u+\mathrm{i}/2)) > 0$, for all α in the line $\Im(z) = 1/2$ there exist T_0 big enough such that $\pi\beta < |\arg(-\psi_X(u+\mathrm{i}/2)T_0^\beta)|$, so that for $T > T_0$ the Stokes lines are not crossed. The correct expression is thus the second line in (7.15), and we can repeat what argued in the DRD case. □

Remark 7.2. The second term in (7.6) is clearly negligible for large T compared to the leading order, when β is smaller than 1. However for fixed T, as β approaches to one its contribution cannot be neglected. This term has been included to clarify the convergence to the Lévy model. Such correction is not present in the SL model and as $\beta = 1$ the price approximation simply breaks down (however, by dominated convergence we still have convergence of prices).

Thus the promised slower convergence of Call prices compared to Lévy (or exponentially-affine stochastic volatility) models. As already remarked, the above proposition can be thought of as a direct consequence of the slow, subdiffusive time spread of the asset returns. More specifically, the nature of the distribution implies that the pricing integral does not obey the Laplace decay rate, since the integrand is not of the form $\exp(-Tf(x))g(x)$. One instead obtains a vanishing long-term volatility, and hence by Lemma 7.1 a persistent long term skew, as we illustrate below:

Corollary 7.3. *For $\beta \in (0,1)$, the leading-order asymptotic for large T of the implied volatility in both the DRD and SL model satisfies*

$$\sigma_\beta(K,T) \sim 2\sqrt{\frac{1}{T}W_0\left(\frac{2KT^{2\beta}\,\Gamma(1-\beta)^2}{\kappa_\beta^2\pi}\right)}, \tag{7.16}$$

where W_0 is the Lambert function and $\kappa_\beta > 0$. Furthermore

$$\lim_{T\to\infty}\frac{T^{-\alpha}}{\mathcal{S}_\beta(K,T)} = 0, \quad \lim_{T\to\infty}\frac{\mathcal{S}_\beta(K,T)}{\sqrt{T}} = 0 \tag{7.17}$$

for all K, $\alpha > 1/2$.

Proof. The first-order expansion of the Black Scholes price is simply

$$C_{\mathrm{BS}}(K,T,\sigma) = 1 - 4e^{k/2}\frac{\exp\left(-\frac{\sigma^2T}{8}\right)}{\sigma\sqrt{2\pi T}}\left(1+\mathcal{O}\left(\frac{1}{T}\right)\right); \tag{7.18}$$

equating this to (7.6) the leading term yields the relation

$$\exp\left(-\frac{\sigma^2T}{8}\right)\frac{4e^{k/2}}{\sigma\sqrt{2T\pi}} = \frac{\kappa_\beta^2}{\Gamma(1-\beta)}T^{-\beta} \tag{7.19}$$

where κ_β is one of the constants C_β or c_β in Proposition 7.2. Setting $z = \sigma^2T/4$, $M = \sqrt{2}e^{k/2}\Gamma(1-\beta)/(\kappa_\beta\sqrt{\pi})$, $w = M^2T^{2\beta}$, then the equality (7.19) reads $e^z z = w$. Since $w > 0$ the inversion in z can be performed along the real axis so that W_0 is well-defined, and (7.16) follows. Since $W_0(T) \sim \log(T)$ as T tends to infinity, then

$$\sigma_\beta(K,T) \sim 2\sqrt{\frac{\log(M^2T^{2\beta})}{T}}, \tag{7.20}$$

therefore $T^{\alpha}\sigma_{\beta}(K,T)$ converges to zero for all $\alpha < 1/2$, which means that the first term of (7.2) tends to zero slower than $T^{-\alpha}$, for all $\alpha > 1/2$, but faster than $T^{-\alpha}$.

Studying the asymptotics of the last term in (7.2), similar arguments to those of Proposition 7.2 imply that the long-term price decay for the Digital option $I_{\{S_T \geq K\}}$ is identical to that of the Call option, namely c/T^{β} for some $c > 0$. Then substituting (7.20) together with $d(x) \sim x/2$, in the second term of (7.6) we have the asymptotic equivalence

$$\frac{\mathbb{Q}(S_T \geq K)}{\sqrt{T}n(d(\sqrt{T}\sigma_{\beta}(K,T)))} \sim c\frac{\exp\left(\frac{T\sigma_{\beta}(K,T)^2}{8}\right)}{T^{\beta+1/2}} = c\frac{\exp\left(\frac{\log(M^2T^{2\beta})}{2}\right)}{T^{\beta+1/2}} = \frac{cM}{\sqrt{T}} \tag{7.21}$$

which completes the proof. □

In light of the corollary above persistence of the skew is to be interpreted as follows: the skew declines slower than any power of T^{-1} bigger than $1/2$ (thus in particular, slower than $1/T$) but always faster than $T^{-1/2}$.

It is then natural to ask if these structural differences in the implied volatility of anomalous diffusion models manifest themselves in the small-maturity limit. It turns out not to be the case, at least for the DRD model, and the underlying Lévy model asymptotics are instead maintained. More precisely, we have the following for Digital option prices.

Proposition 7.4. *If the underlying Lévy process X is such that*

$$\mathbb{Q}(S_t^0 \geq 1) \sim c_0 + c_{\varepsilon}t^{\varepsilon} + o(t^{\varepsilon}), \tag{7.22}$$

for some c_0, c_{ε}, as t tends to zero, with $0 < \varepsilon \leq \frac{1}{2}$, then, with $c_{\beta,\varepsilon} := \frac{\Gamma(\beta+\varepsilon)}{\Gamma(\beta)\Gamma(1+\varepsilon)}$,

$$\mathbb{Q}\left(S_t^{DRD} \geq 1\right) \sim c_0 + c_{\beta,\varepsilon}c_{\varepsilon}t^{\varepsilon} + o(t^{\varepsilon}). \tag{7.23}$$

Proof. Proposition 4.3 allows us to write

$$\mathbb{Q}\left(Y_t^{DRD} \geq 0\right) = \int_0^t \mathbb{Q}(X_s \geq 0)\frac{s^{\beta-1}(t-s)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)}ds. \tag{7.24}$$

Now, notice that $\mathbb{E}\left[\mathcal{B}^a_{\beta,1-\beta}\right] = \frac{\Gamma(\beta+a)}{\Gamma(\beta)\Gamma(1+a)}$ for all $a > 0$, and that for t sufficiently small,

$$\mathbb{Q}(X_t \geq 0) = c_0 + c_{\varepsilon}t^{\varepsilon} + f(t), \tag{7.25}$$

where $f(t) = o(t^{\varepsilon})$ is a bounded function in a neighbourhood of the origin. The zero- and first-order terms of (7.23) are then clear, and by dominated convergence:

$$\lim_{t\to 0}\mathbb{E}[f(t\mathcal{B}_{\beta,1-\beta})]t^{-\varepsilon} = \int_0^1 \lim_{t\to 0}\frac{f(ts)}{t^{\varepsilon}}\frac{s^{\beta-1}(1-s)^{-\beta}}{\Gamma(\beta)\Gamma(1-\beta)}ds = 0, \tag{7.26}$$

which yields the small-o order ε of the remainder. □

Equation (7.22) essentially encompasses all the popular Lévy models, and features very different behaviours: for example $c_0 = 1$ if the process has finite variation, whereas $c_0 = 1/2$ and $c_\varepsilon = 1/2 + d/(\sigma\sqrt{2\pi})$, $\varepsilon = 1/2$ for a jump diffusion with volatility σ and risk-neutral drift d. There is a stringent relationship between the prices of Digital options and the small-time at-the-money skew, made precise by (7.3), and the critical value $c_0 = \frac{1}{2}$ for which higher-order terms are needed. For a full account and more details we refer to Gerhold et al. (2016). In the DRD model, introducing $c_{\beta,\varepsilon}$ does not change the asymptotic analysis, as c_0 remains the same.

Corollary 7.5. *If X satisfies* (7.22)*, then the DRD model and the underlying exponential Lévy model S^0 have the same short-maturity at-the-money skews.*

In the next section we bring together all these results and see how they lead to model calibration improvements when a persistent implied volatility skew is observed.

8 Numerical analysis

8.1 Volatility skew and term structure

We visualize the volatility surfaces extracted from models DRD and SL in Figures 4 to 7. For X, we use a Brownian motion (Figures 4 and 5) and a CGMY process with parameters taken from Carr. et al. (2001) (Figures 4 and 6), and consider moneynesses $\pm 40\%$ ATM and maturities up to two years. In each figure, the smile of the anomalous diffusion is compared to that of its underlying Lévy model S^0.

First and foremost the slower decay of the skew with maturity of the anomalous diffusion model compared to the underlying Lévy model is clearly apparent in all cases. At least in the DRD case, even though Proposition 7.2 and Corollary 7.3 only predict an asymptotic rate of skew vanishing, our numerical tests indicate that the rate manifests itself for already very early on and for a wide range of maturities. More research is necessary to see whether and how Proposition 7.2 can be improved.

In Figures 4 and Figures 5 the volatility smile and skew for the anomalous diffusion model are present even if the Lévy generating returns process is a Brownian motion. In order words, this confirms that introducing infinite-mean trade durations in a standard CTRW is alone sufficient to generate a smile, consistently with Proposition 5.2. The smile appears rather symmetric, in line with the intuition that trade duration should have little skew impact, as it does not influence out-of-the-money prices any differently than in-the-money ones. This already suggests some orthogonality between β and the Lévy parameters. In the Brownian motion case, β is thus 'overloaded', being responsible for both the smile convexity and its decay rate. This is relaxed in 6 and Figures 7 by endowing X with a proper Lévy structure (CGMY); there a short-term skew arises while the skew term structure maintains its slower flattening rate, dictated by β.

In Figures 4 and 6 we observe the repercussion on the implied volatilities of the 'cross-over' phenomenon (Figure 2) generated by the Mittag-Leffler and exponential types of the characteristics functions of the SL and pure Lévy models. The level of the SL surfaces transitions from a short-term regime where the implied volatilities are higher to a long-term one in which they are lower than those of the underlying Lévy models (eventually tending to zero). Such a transition seems to be very sharp.

The time sections from Figures 6 and 7 are shown in Figures 8 and 9 and further highlight the remarks above. Figures 10 to 13 highlight the convergence of the time sections to those of the underlying Lévy model as β approaches one. For the SL model this convergence is from above, while it is from below for the DRD model. Note also that the DRD model exhibits a sharper ATM skew than the SL model.

8.2 Calibration

Corollary 7.3 and Proposition 7.4 suggest that, from a calibration viewpoint, the models should behave as follows: the Lévy parameters have a short-time scale effect, unaffected when introducing β, and they should hence absorb the short-time skew and smile. however, β is the very component governing the long-term structure of the surface, where the Lévy structure is flat and has no impact, and should thus allow to pick up the long-term skew. To test this we generate 3-month and 6-month volatility skews from a given Lévy model S^0, which represent our baseline synthetic market data. In order to generate two scenarios of persistent volatility skew, while keeping the 3-month fixed, we shift the 6-month skew forward to make it coincide respectively with the 1-year and 18-month skews. We then cross-sectionally calibrate S^0, S^{SL} and S^{DRD} to the 3-month and 6-month skews in the baseline scenario, and the 3-month and 1-year (respectively 18-month) sections in the first and second scenarios.

The calibration has been performed using a Differential Evolution global optimizer. Let $C(K,T;\beta,\Gamma)$ be the theoretical Call prices from the SL or DRD models, where Γ denotes the set of the Lévy parameters for X, and by $C(K,T)$ the synthetic market prices obtained with the procedure described above. We use the in-price norm, and minimize the total squared errors

$$\arg\min_{\beta,\Gamma} \sum_{i,j} \left| C(K_i,T_j;\beta,\Gamma) - C(K_i,T_j) \right|^2 . \tag{8.1}$$

The base process X is taken as a Normal Inverse Gaussian (NIG; Barndorff-Nielsen 1997) and a Variance Gamma (VG; Madan et al. 1998), and we show the results in Tables 1 to 3. In Table 1 we represent the baseline scenario: all three models perfectly fit the synthetic Lévy market data. Correctly, the β parameter in the SL and DRD models calibrates to one, and produces no improvement on the S^0 calibration. In the scenarios with a persistent long-term skew, the total error for the Lévy model is greater than that for the SL and DRD models, with $\beta < 1$. Comparing the two scenarios we observe as expected that for both models, β is less in the second scenario than in the first one, owing to a steeper long-term skew in the second case. This can be interpreted as an asset with a more prolonged trade duration.

Comparing errors across the models, the SL model shows a better fit in all cases. However this should not necessarily be interpreted as an overall superiority: the better calibration might be only due to the synthetic market data generated by a Lévy model, and the SL distributions being closer to Lévy.

9 Conclusion

We have proposed the use of anomalous diffusion processes in the context of option pricing, which allows to naturally incorporate trade durations between price moves. Using limits of CTRWs whose inter-arrival times distribution obeys a power law to model asset returns, we analysed the impact on the term structure of the returns distribution and on the corresponding implied volatility. More specifically, the observed volatility skew persistence on the market can be explained by a non-negligible impact of trade time randomness even in the long-term price evolution.

We analysed both cases when the price innovations are dependent and are independent from the waiting times between trades. Both models are consistent with the econometric observation that shorter duration generates sharper variations in the price revisions. Finally, we remarked that even though the two models lead to similar large-maturity implied volatility properties, their different distributional properties produce rather different shapes of volatility surfaces. Numerical experiments confirm that for option pricing anomalous diffusions models have the potential to capture the slow decay of the volatility skew while retaining the short-term good properties of pure Lévy models.

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10 Tables and Figures

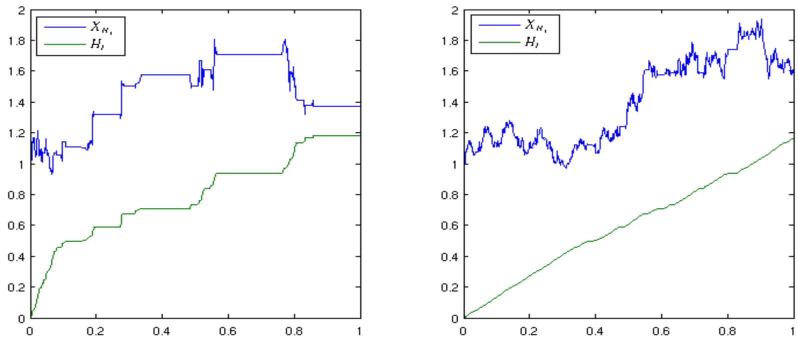

Figure 1: Paths of X_H (blue) and H (green) in the SL model. $\beta = 0.7$ on the left and $\beta = 0.95$ on the right. Here, X is a driftless Brownian motion with diffusion parameter $\sigma = 0.4$.

Parameter	Lévy		SL		DRD	
	VG	NIG	VG	NIG	VG	NIG
κ	0.2037	0.2822	0.2037	0.2828	0.2037	0.2845
σ	0.3002	0.1994	0.3002	0.1989	0.3002	0.1995
θ	-0.2983	-0.1039	-0.2983	-0.1036	-0.2983	-0.1039
β	-	-	1.0000	0.9977	1.0000	0.9995
Error	0.0000	0.0003	0.0000	0.0003	0.0000	0.0003

Table 1: Calibration to the 1-month Lévy smile generated by the base model S^0. The parameters are (κ, σ, θ) are $(0.2, 0.3, -0.3)$ for the VG model and $(0.3, 0.2, -0.1)$ for the NIG model.

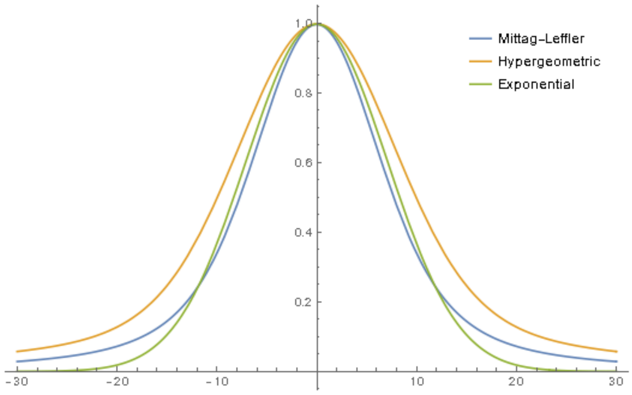

Figure 2: Comparison of the function Φ_t for the SL and DRD model with the exponential, $\beta = 0.75$, $t = 0.5$. We used the compensated geometric Brownian motion characteristic exponent $\phi_X(z) = \sigma^2(z^2 - \mathrm{i}z)/2$ along the line $\Im(z) = 1/2$ where it is real.

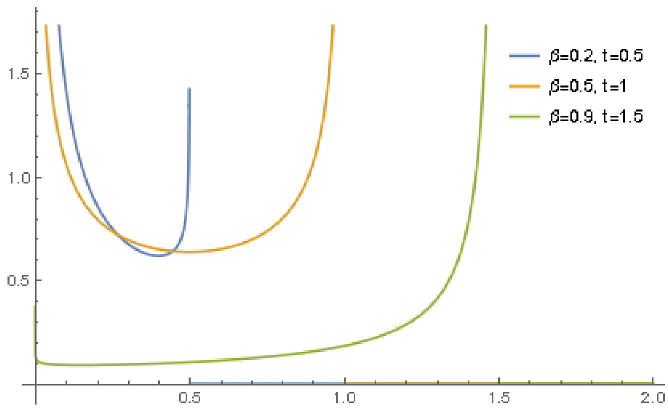

Figure 3: Densities of the time change $L_t^H \sim t\mathcal{B}_{\beta,1-\beta}$. For each t the total integral at some value x has the interpretation of the probability that the time for the background Lévy process X ran at most up to x.

Parameter	Lévy		SL		DRD	
	VG	NIG	VG	NIG	VG	NIG
κ	1.4474	7.6080	1.5482	6.7625	0.8707	2.3789
σ	0.3298	0.2635	0.3218	0.2525	0.3785	0.2988
θ	-0.1696	-0.0556	-0.1739	-0.0546	-0.2810	-0.0938
β	-	-	0.8669	0.8837	0.7164	0.6602
Error	0.1355	0.0425	0.0703	0.0299	0.0863	0.0318

Table 2: Calibration to the 1-month and 1-year shifted Lévy smile generated by the base model S^0. The parameters (κ, σ, θ) are $(0.2, 0.3, -0.3)$ for VG and $(0.3, 0.2, -0.1)$ for NIG.

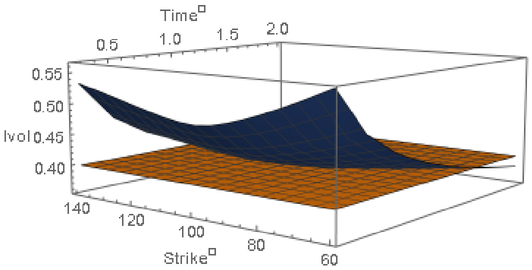

Figure 4: SL implied volatility surface based on geometric Brownian motion; $\sigma = 0.4$, $\beta = 0.7$.

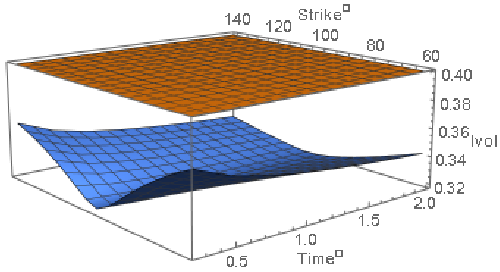

Figure 5: DRD implied volatility surface from geometric Brownian motion; $\sigma = 0.4$, $\beta = 0.7$.

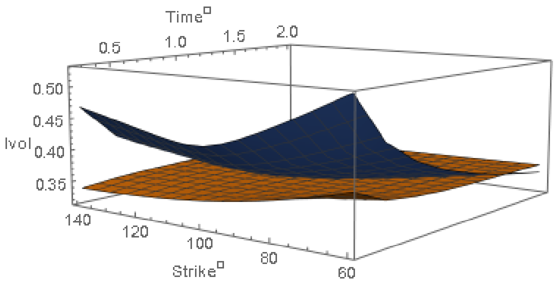

Figure 6: SL implied volatility surface based on a CGMY Lévy model, with $C = 6.51, G = 18.75, M = 32.95, Y = 0.5757, \beta = 0.7$.

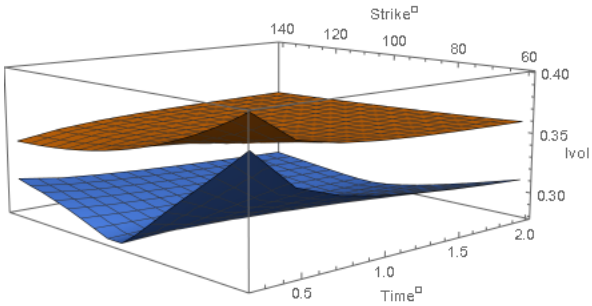

Figure 7: DRD implied volatility surface based on a CGMY Lévy model, with $C = 6.51, G = 18.75, M = 32.95, Y = 0.5757, \beta = 0.7$.

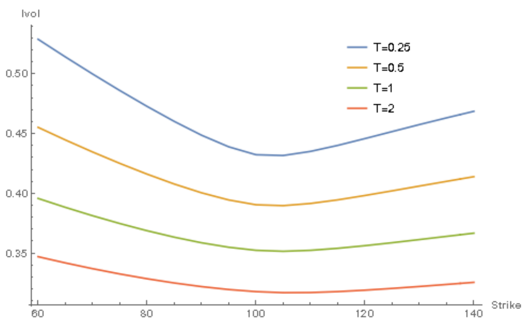

Figure 8: Time sections from Figure 6.

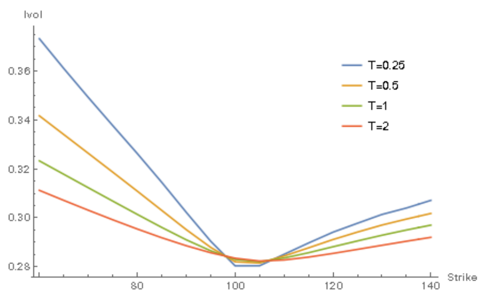

Figure 9: Time sections from Figure 7.

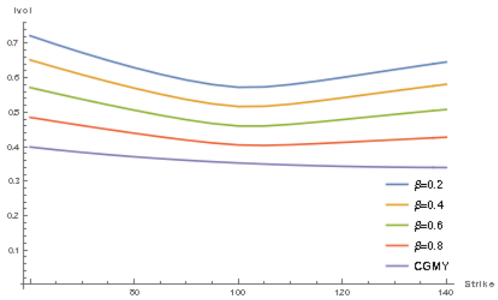

Figure 10: Convergence of the SL skew to the CGMY one as β tends to one, with $T = 0.25$.

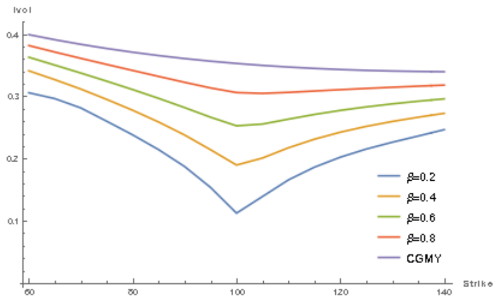

Figure 11: Convergence of the DRD skew to the CGMY one as β tends to one, with $T = 0.25$.

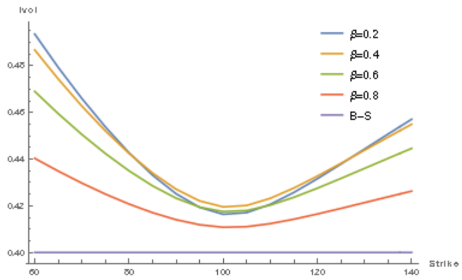

Figure 12: Convergence of the SL model to the BS volatility as β tends to one, for $T = 0.75$.

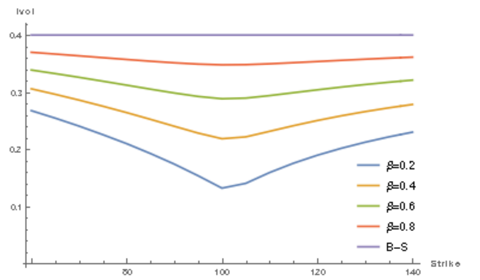

Figure 13: Convergence of the DRD skew to the BS volatility as β tends to one, for $T = 0.75$.

Parameter	Lévy		SL		DRD	
	VG	NIG	VG	NIG	VG	NIG
κ	4.5443	42.5059	3.2555	30.5834	2.0072	9.3440
σ	0.3952	0.4022	0.3661	0.3404	0.4562	0.4022
θ	-0.1354	-0.0785	-0.1571	-0.0711	-0.2534	-0.1138
β	-	-	0.8305	0.8634	0.6370	0.5626
Error	0.2359	0.0732	0.1305	0.0532	0.1738	0.0586

Table 3: Calibration to 1-month and 18-month shifted Lévy smiles generated by S^0. The parameters (κ, σ, θ) are $(0.2, 0.3, -0.3)$ for the VG model and $(0.3, 0.2, -0.1)$ for the NIG model.